Logic for CS

Notes for Ben's Course at UWaterloo[1]

[1]https://www.youtube.com/playlist?list=PLPW2keNyw-utXOOzLR-Wp1poeE5LEtv3N

Contents

.1 Propositional Logic

.1.1 Proof System

Definition 1 (consistent). *A set of w.f.f.* Σ *is consistent if* $\Sigma \vdash \alpha$ *and* $\Sigma \vdash \neg\alpha$ *does not hold together.*

Another definition of consistency is that $\Sigma \nvdash \bot$ where $\bot$ is a contradiction[a].

[a] https://planetmath.org/consistent

Claim 1. *A set of w.f.f.* Σ *is consistent iff* $\exists\alpha, \Sigma \nvdash \alpha$.

Proof. $\rightarrow$: pick any α, either $\Sigma \nvdash \alpha$ or $\Sigma \vdash \alpha$, then there exists one that is not proved by Σ.

$\leftarrow$: assume that for some $beta$, $\Sigma \nvdash \beta$, if the first definition is violated, then, for some α, $\Sigma \vdash \alpha$ and $\Sigma \vdash \neg\alpha$, then $\Sigma \vdash \Sigma \cup \{\alpha, \neg\alpha\} \vdash \{\alpha, \neg\alpha\}$. $\{\alpha, \neg\alpha\} \vdash \beta$ for every β, contradicting the assumption. □

Corollary 1. *If* $\Sigma \subseteq \Sigma'$*, then if* Σ *is consistent, then so does* Σ'.

Definition 2 (Soundness). *If* $\vdash \alpha$*, then* α *is a tautology.*

In another word, every theorem is a tautology.

Corollary 2. $\nvdash P$*, since* P *is a propositional variable, which is not a tautology.*

The universe U (the set which includes everything) is the least consistent, since it contains everything. Therefore, for every $\alpha \in U$, $\neg\alpha \in U$, U can prove both. It has the most unsteady stance:).

$\vdash$ is about syntax while $\vDash$ is about semantics. $\vdash$ shows that though the machine doesn't know anything, it can reach some result by axioms and modus ponens. $\vDash$ shows that the result is true in the real world given by human being's assignment of truth values.

Theorem 1. *In a sound proof system, every satisfiable Σ is consistent.*

Proof. *b.w.o.c*, assume the satisfiable Σ is inconsistent. Then, for some α, both $\Sigma \vdash \alpha$ and $\Sigma \vdash \neg\alpha$. If Σ is satisfiable, then for some truth assignment V, V satisfies all *w.f.f.*in Σ. By soundness, $\Sigma \vDash \alpha$ and $\Sigma \vDash \neg\alpha$. So for that assignment V, we get $V(\alpha) = T$ and $V(\neg\alpha) = T$, violating the truth table of negation. □

The above theorem builds a bridge from semantic to syntax.

Theorem 2 (Extended Soundness)**.** *If* $Sigma \vdash \alpha$*, then* $\Sigma \vDash \alpha$*.*

Corollary 3. *If any set of w.f.f.is consistent, then, in particular* $\emptyset$*.*

Definition 3 (Maximally Consistent)**.** *We say that* Σ *is maximally consistent if* Σ *is consistent, but, for every* α*, either* $\Sigma \vdash \alpha$ *or* $\Sigma \cup \{\alpha\}$ *is inconsistent.*

Consistency tells us that if $\Sigma \vdash \alpha$, then $\Sigma \nvdash \alpha$ doesn't hold. But if $\Sigma \nvdash \alpha$, then we can't say $\Sigma \vdash \alpha$ holds. Consistency only ensures no contradiction.

Maximally consistency tells us that if $\Sigma \nvdash \alpha$, then $\Sigma \vdash \alpha$ must hold. The maximality nature ensures that the negation of every nonprovable *w.f.f.*is provable.

Example 1. *Let* $\Sigma \equiv \{P_1\}$ *over the variables* $P_1, P_2, \ldots$*.*

Claim 2. $\{P_1\}$ *is consistent since it's satisfiable.*

Claim 3. $\{P_1\}$ *is not maximally consistent*

Proof. It suffices to show that $\{P_1\} \nvdash P_3$ and $\{P_1, P_3\}$ is consistent.

By soundness, it suffices to show that $\{P_1\} \vDash P_3$. Just find some truth assignment that V *s.t.*$V(P_1) = T$ and $V(P_3) = F$. So $\{P_1\} \nvdash P_3$.

Since $\{P_1, P_3\}$ is satisfiable, so it's consistent. □

Lemma 1. *For every consistent Σ, there exists a maximally consistent $\Sigma' \supseteq \Sigma$.*

Proof. Let $\{\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots\}$ is a set of *w.f.f.*over $\{P_1, P_2, \ldots, P_i, \ldots\}$ □

Definition 4 (Monotonicity). $\forall \Sigma, \Sigma', \alpha$, *if* $\Sigma \vdash \alpha, \Sigma \subseteq \Sigma'$, *then* $\Sigma' \vdash \alpha$.

Theorem 3 (Deduction Theorem). $\Sigma \cup \{\alpha\} \vdash \beta$ *iff* $\Sigma \vdash \alpha \to \beta$.

Theorem 4. *Any consistent set of w.f.f.Σ can be extended to Σ's.t.$\Sigma' \supseteq \Sigma$ that is maximally consistent.*

Proof. We construct a sequence of set of *w.f.f.*. $\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \ldots \subseteq \Sigma_i \subseteq \Sigma_{i+1}$ *s.t.*1) $\Sigma_0 = \Sigma$ and 2) For all Σ_i, Σ_i is consistent, and 3) for fixed enumeration of all *w.f.f.*, $\alpha_1, \alpha_2, \ldots, \alpha_n$, for all i, either $\Sigma_i \vdash \alpha$ or $\Sigma_i \vdash \neg\alpha$.

Assume Σ_i is defined and meets the requirements, let Σ_{i+1} be Σ_i if $\Sigma_i \vdash \neg\alpha_i$, otherwise $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_i\}$.

Now we want to prove if Σ_i meets the requirement, then so will Σ_{i+1}. Requirement 1) is trivial. Requirement 2) and 3) are simultaneously proved by showing $\Sigma_{i+1} \nvdash \neg\alpha_i$ if $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_i\}$, which is shown by the following claim.

Claim 4. $\forall \Sigma, \alpha$, *if* $\Sigma \cup \{\alpha\} \vdash \neg\alpha$, *then* $\Sigma \vdash \neg\alpha$.

Proof. To prove this claim, it suffices to show $\vdash (\alpha \to \neg\alpha) \to \neg\alpha$ based on deduction theorem, which is a tautology. □

Now we show the $\Sigma' = \cup_{i=1}^{\infty} \Sigma_i$ is maximally consistent. To prove it, it suffices to show that $\forall \alpha_i, i \in N, \Sigma' \vdash \alpha$ or $\Sigma' \vdash \neg\alpha$. By requirement 2), $\Sigma_i \vdash \alpha_i$ or $\Sigma_i \vdash \neg\alpha_i$. Now we are going to show $\Sigma' \vdash \alpha_i$ or $\Sigma' \vdash \neg\alpha_i$.

b.w.o.c, suppose Σ' is inconsistent. In this case, for some α, $\Sigma' \vdash \alpha\, and\, \Sigma' \vdash \neg\alpha$. Let $\beta_1, \ldots \beta_k$ be the proof of α from Σ' and $\gamma_1, \ldots, \gamma_l$ be the proof of $\neg\alpha$ from Σ'.

Each β_i that is an assumption from Σ' belongs to some Σ_{m_i}, and each γ_i that is an assumption from Σ' belongs to some Σ_{m_j}. Since both formal proofs of α and of $\neg\alpha$ are finite, there is some i^* that is bigger than all of these m_i's and m_j's. Therefore, for each $beta_i$ or $gemma_j$ that are used as assumptions, $\beta_i, \gamma_j \in \Sigma_{i^*}$. Now by monotonicity, $\Sigma_{i^*} \vdash \alpha$ and $\Sigma_{i^*} \vdash \neg\alpha$, which is a contradiction to requirement 2).

□

Theorem 5. *In a sound proof system, every consistent Σ is satisfiable.*

Also called the completeness theorem.

Proof. Let Σ' be a maximally consistent set of *w.f.f.*.s *s.t.* $\Sigma \subseteq \Sigma'$. Define a truth assignment $V_{\Sigma'}$ as follows: $V_{\Sigma'}(P_i) = T$ iff $\Sigma' \vdash P_i$ where P_i is a propositional variable. Now extend it into $V_{\Sigma'}(\alpha_i) = T$ iff $\Sigma' \vdash \alpha_i$.

Claim 5. *For every formula α, $\overline{V}_{\Sigma'}(\alpha) = T$ iff $\Sigma' \vdash \alpha$, where $\overline{V}_{\Sigma'}$ is the extension of $V_{\Sigma'}$ in the infinite set of w.f.f., and $\overline{V}_{\Sigma'}$ is the truth assignment that satisfies Σ'.*

Proof. By generalized induction on the set of all *w.f.f.*: $I(\mathcal{P}, \{\rightarrow, \neg\})$ where $\mathcal{P}$ is the set of propositional variables, and $\{\rightarrow, \neg\}$ are the adequate connectives (adequate means that the set of connectives are enough to map all truth assignment to the truth result). $I(\mathcal{P}, \{\rightarrow, \neg\})$ means that all *w.f.f.*s that are built from $\mathcal{P}$ and $\{\rightarrow, \neg\}$.

Induction base. $\alpha \in \mathcal{P}$.

$\leftarrow$: $\overline{V}_{\Sigma'}(\alpha) = T$, then $\Sigma' \vdash \alpha$ by definition of $V_{\Sigma'}$.

$\rightarrow$: Suppose $\Sigma' \nvdash \alpha$, then $\overline{V}_{\Sigma'}(\alpha) = F$ still by the definition.

Remember the relation between truth assignment and the proof symbol. $\Sigma \vdash \alpha$ means that for every truth assignment V

s.t. $V(\Sigma) = T$, then $V(\alpha) = T$, which can be simplified as $V_\Sigma(\alpha) = T$.

Induction step. Assume the claim holds for α and for β, we need to show that for $\neg\alpha$ and $\alpha \to \beta$.

$\neg\alpha$:

$\leftarrow$: If $\Sigma' \vdash \neg\alpha$, by the consistency, $\Sigma' \nvdash \alpha$. So by the induction hypothesis, $\overline{V}_{\Sigma'}(\alpha) = F$, so by the truth table, $\overline{V}_{\Sigma'}(\neg\alpha) = T$.

$\to$: Assume $\Sigma' \nvdash \neg\alpha$, then by its maximality, $\Sigma' \vdash \alpha$. So by the induction hypothesis, $\overline{V}_{\Sigma'}(\alpha) = T$, so by the truth table, $\overline{V}_{\Sigma'}(\neg\alpha) = F$.

$\alpha \to \beta$:

$\leftarrow$: $\Sigma' \nvdash (\alpha \to \beta)$, Since $\vdash \beta \to (\alpha \to \beta)$, $\Sigma' \nvdash \beta$. By the induction hypothesis, $\overline{V}_{\Sigma'}(\beta) = F$. Now we need to show that $\overline{V}_{\Sigma'}(\alpha) = T$. *b.w.o.c*, assume $\overline{V}_{\Sigma'}(\alpha) = F$. By the induction hypothesis, $\Sigma \nvdash \alpha$, and by the maximality of Σ', $\Sigma' \vdash \neg\alpha$. And by $\vdash \neg\alpha \to (\alpha \to \beta)$, $\Sigma' \vdash (\alpha \to \beta)$, which contradicts $\Sigma' \nvdash (\alpha \to \beta)$. Therefore, $\overline{V}_{\Sigma'}(\alpha) = F$, so this case is proved.

$\to$: $\Sigma' \vdash (\alpha \to \beta)$. Assume $\Sigma' \vdash \alpha$, in which case $\Sigma' \vdash \beta$ by modus ponens. Using the induction hypothesis, $\overline{V}_{\Sigma'}(\alpha) = T$ and $\overline{V}_{\Sigma'}(\beta) = T$, so $\overline{V}_{\Sigma'}(\alpha \to \beta) = T$. Otherwise, $\Sigma' \nvdash \alpha$, so by the induction hypothesis, $\overline{V}_{\Sigma'}(\alpha) = F$, so by truth table, $\overline{V}_{\Sigma'}(\alpha \to \beta) = T$. □

□

Proof. Σ is satisfiable by all-truth assignemnt, so it's consistent. To show it's maximally consistent, we need to show for every α, if $\Sigma \nvdash \alpha$, then $\Sigma \cup \{\alpha\}$ is inconsistent. If $\Sigma \nvdash \alpha$, then $\Sigma \nvDash \alpha$, then $\Sigma \cup \{\alpha\}$ is unsatisfiable, so it's inconsistent.

Based on the definition of the truth table, $V_\Sigma = T$. □

Lemma 2. *If* $\Sigma \nvdash \alpha$ *and* Σ *is consistent, then* $\Sigma \cup \{\neg\alpha\}$ *is consistent.*

Proof. Suppose $\Sigma \cup \{\neg\alpha\}$ is inconsistent and Σ is consistent, then $\exists\beta, \Sigma \cup \{\neg\alpha\} \vdash \beta$ and $\Sigma \cup \{\neg\alpha\} \vdash \neg\beta$. By deduction theorem,

$\Sigma \vdash \neg\alpha \to \beta$ and $\Sigma \vdash \neg\alpha \to \neg\beta$. Since Σ is consistent, $\Sigma \vdash \beta$ and $\Sigma \vdash \neg\beta$ do not hold together.

Assume $\Sigma \vdash \beta$, from $\Sigma \vdash \neg\alpha \to \neg\beta$, we know $\Sigma \vdash \beta \to \alpha$, so $\Sigma \vdash \alpha$, contradicting the assumption that $\Sigma \nvdash \alpha$.

Assume $\Sigma \nvdash \beta$, from $\Sigma \vdash \neg\alpha \to \beta$, we know $\Sigma \vdash \neg\beta \to \alpha$, so $\Sigma \vdash \alpha$, contradicting the assumption that $\Sigma \nvdash \alpha$.

Since Σ is not maximally consistent, so Σ may not prove either β or $\neg\beta$. In this situation, β is α or $\neg\alpha$ and obviously $\Sigma \cup \{\neg\alpha\}$ can only prove either β or $\neg\beta$, contraicting $\Sigma \cup \{\neg\alpha\} \vdash \beta$ and $\Sigma \cup \{\neg\alpha\} \vdash \neg\beta$. □

Theorem 6 (Completeness). *For all α and any set of w.f.f.s Σ, if Σ is consistent and $\Sigma \models \alpha$, then $\Sigma \vdash \alpha$.*

Proof. Suppose $\Sigma \nvdash \alpha$, then $\Sigma \cup \{\neg\alpha\}$ is consistent. Therefore, $\Sigma \cup \{\neg\alpha\}$ is satisfiable, so $\Sigma \not\models \alpha$. □

Corollary 4. *Σ is maximally consistent, iff it is maximally satisfiable.*

Proof. $\to$: Since every α, $\Sigma \vdash \alpha$ or $\Sigma \vdash \neg\alpha$, by soundness theorem, $\Sigma \models \alpha$ or $\Sigma \models \neg\alpha$.

$\leftarrow$: Since for every α, $\Sigma \models \alpha$ or $\Sigma \models \neg\alpha$, we get $\Sigma \vdash \alpha$ or $\Sigma \vdash \neg\alpha$ by completeness theorem. □

> Is there a polynomial algorithm to figure out a give α is satisfiable?
>
> Does there exist a proof P_f of α such that $|P_f| < Poly(|\alpha|)$ where $Poly$ is a polynomial function.
>
> Does there exist any sound and complete proof systemw with the above property?